Lecture Notes for Abstract Algebra: Lecture 8

1 Lagrange theorem

1.1 Cosets and Lagrange Theorem

Definition 1. Let $(G, *)$ be a group. Let $H \leqslant G$ be a subgoup. The left coset of an element $x \in G$ is the set $xH = \{x * h \,|\, h \in H\}$. The right coset of an element $x \in G$ is the set $Hx = \{h * x \,|\, h \in H\}$.

Example 2. Consider the subgroup $K = \langle(1\,2)\rangle = \{(1), (1\,2)\}$ of S_3. Then the left cosets of K are:

$$(1)K = (12)K = \{(1), (12)\},$$
$$(13)K = (123)K = \{(13), (123)\},$$
$$(23)K = (132)K = \{(23), (132)\}.$$

The right cosets on the other hand are:

$$K(1) = (12)K = \{(1), (12)\},$$
$$K(13) = K(132) = \{(13), (132)\},$$
$$K(23) = K(123) = \{(23), (123)\}.$$

Lemma 3. *Let H be a subgroup of a group G and suppose that $g_1, g_2 \in G$. The following conditions are equivalent.*

(1) $g_1H = g_2H$.

(2) $g_1^{-1}g_2 \in H$.

(3) $Hg_1^{-1} = Hg_2^{-1}$.

(4) $g_2 \in g_1H$.

Definition 4. Let G be a group. Let $H \leqslant G$ be a subgoup. The subgroup H determines an equivalence relation on G given by equality of left cosets

$$x \sim y \iff xH = yH.$$

The quotient set $G/\sim_H$, is the set $\{xH \,|\, x \in G\}$ of left cosets.

Definition 5. Let $H \leqslant G$ be a subgoup, the index of H in G, denoted by $[G : H]$, is the cardinality of the set of left cosets $\{xH \,|\, x \in G\}$.

Proposition 6. *Let H be a subgroup of a group G. The number of left cosets of H in G is the same as the number of right cosets of H in G.*

Proof. Let L_H and R_H denote the set of left and right cosets of H in G, respectively. We can define a bijective map

$$\Phi\colon L_H \longrightarrow R_H$$

by the formula $\Phi(gH) = Hg^{-1}$. The map is well defined and bijective because of lemma 3. □

Corollary 7. *Let H be a subgroup of G. If the index of $[G : H] = 2$, the left and right cosets are the same.*

Theorem 8. *(Lagrange Theorem) Let G be a group and $H \leqslant G$ a subgroup. Then*

1. *Let $x, x' \in H$, any two left cosets, xH and $x'H$, has the same cardinality.*
2. *If $|G|$ has finite order, then $|G| = |H|[G : H]$.*

Proof. The map $\psi\colon xH \longrightarrow x'H$ defined by $\psi(x * h) = x' * h$ is bijective. The set G is therefore partitioned in $[G : H]$ equivalence classes of cardinality $|H|$. □

Example 9. Consider the alternate group A_4 of order 12. The subgroups of order 2 are given by:

$$\{\langle(12)(34)\rangle, \langle(13)(24)\rangle, \langle(14)(23)\rangle\}.$$

Now consider the elements $(14)(23), (12)(34), (13)(24)$. We can check that multiplication of any two of those elements, gives the third one, for example:

$$(12)(34) \circ (14)(23) = (13)(24) \qquad (12)(34) \circ (13)(24) = (14)(23)$$

Hence we can make the subgroup of order 4;

$$H = \langle(12)(34), (14)(23)\rangle = \{1, (12)(34), (13)(24), (14)(23)\}$$

On the other hand, the subgroups of order three can be found generated by elements of order three:

$$\{\langle 123\rangle = \langle 132\rangle, \langle 124\rangle = \langle 142\rangle, \langle 134\rangle = \langle 143\rangle, \langle 234\rangle = \langle 243\rangle\}.$$

In general if the index of a group H, $[G : H] = 2$, the group H must contains all elements of odd order. But in A_4 there are 8 elements of order 3. Hence, there is no such subgroup of order 6 in A_4, showing that the converse of Lagrange theorem is, in general, not true.

Order	Subgroups
1	$\langle 1 \rangle = \{1\}$
2	$\{(1), (12)(34)\}; \{(1), (13)(24)\}; \{(1), (14)(23)\}$
3	$\{(1), (123), (132)\}; \{(1), (124), (142)\}$ $\{(1), (134), (143))\}; \{(1), (234)(243)\}$
4	$\{(1), (12)(34), (13)(24), (14)(23)\}$
12	A_4

Corollary 10. *Let G be a finite group and $g \in G$. The order g must divide the order of G.*

Corollary 11. *Let p be a prime number and G a group of order p. Then, the group G must be cyclic generated by any element $g \neq e$ in G.*

Practice Questions:

1. As a group of order four, what type of subgroup is H, a $\mathbb{V}_4$ or a $\mathbb{Z}_4$?

2. Show that if $H \leqslant G$ is a subgroup of index 2, then the group H must contains all elements of odd order.